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# Generic entangled states

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**Abstract.** It is shown how to construct generic entangled states for an arbitrary system of $n$-state quantum objects (qunits) by means of the $(n \times n)$ cyclic permutation operator.

Modern classification of entangled states of composite systems is based on the use of certain local transformations usually called SLOCC (stochastic local operations assisted by classical communications) (Dür *et al* 2000, Klyachko 2002, Miyake 2003 and references therein). Namely, the entangled states are equivalent to each other to within SLOCC. This means, that action of SLOCC on a given entangled state can change the amount of entanglement but cannot transform the state into an unentangled one. In turn, unentangled states cannot be transformed into entangled states by means of SLOCC.

Note that SLOCC are deeply related to the symmetry properties of parties of the composite system (for simplicity, we assume a system consisting of equivalent parties). If $G = \exp(\mathcal{L})$ is the Lie group of dynamic symmetry of the party associated with the Lie algebra $\mathcal{L}$ of local observables, then SLOCC transformations $g$ are defined to be $g \in G$ (Klyachko 2002, Klyachko and Shumovsky 2004).

It follows from above classification that all entangled states of a given system can be constructed from a certain completely entangled *generic state*. For example, all entangled states of two qubits can be constructed from the Bell states

$$|\psi_{Bell}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle). \tag{1}$$

In turn, entangled states of three qubits can be constructed by means of SLOCC from the GHZ (Greenberger-Horne-Zeilinger) state (Miyake 2003)

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle). \tag{2}$$

It was shown by Can *et al* (Can *et al* 2002) that generic entangled states of any number $N$ of qubits are given by the so-called $su(2)$ phase states of dimension two. The latter are defined by the polar decomposition of the $su(2)$ algebra (Vourdas 1990, Shumovsky 2001)

$$J_+ = J_r E, \quad J_- = E^+ J_r, \quad EE^+ = \mathbf{1}, \tag{3}$$

where

$$[J_+, J_-] = 2J_z, \quad [J_z, J_\pm] = \pm J_\pm \tag{4}$$

and $J_r = (J_+J_-)^{1/2}$. Namely, the $su(2)$ phase states $|\phi\rangle$ are the eigenstates of the exponential phase operator $E$:

$$E|\phi_k\rangle = e^{i\phi_k}|\phi_k\rangle, \tag{5}$$

where $\phi_k$ denotes a certain angle.

The aim of this note is to show that generic entangled states of an arbitrary system of $N$ *qunits* are the $su(2)$ phase states of dimension $n$ (also see Binicioglu *et al* 2005). Here by qunit we mean an $n$-state quantum system with the basic lobservables given by the orthogonal basis of the $su(n)$ algebra. The most popular examples are provided by qubits with $n = 2$ and qutrits with $n = 3$. In the first case, basic observables are given by the three spin-projection operators (Pauli matrices). In the latter case basic observables are eight independent Hermitian generators of the $su(3)$ algebra (Caves and Milburn 2000).

For a *qunit* with an arbitrary $n$, the exponential phase operator $E$ has the form of $(n \times n)$ cyclic permutation operator

$$E = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{i\varphi} & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{6}$$

where $\varphi$ is an arbitrary reference phase. Below we put $\varphi = 0$.

It is easily seen that in the case of two qubits the $su(2)$ exponential phase operator can be defined as follows

$$E_{2,2} = |00\rangle\langle 11| + |11\rangle\langle 00|$$

and that its eigenstates are just the Bell states (1). This result can be immediately generalized on the case of an arbitrary number $N$ of qubits:

$$E_{2,N} = |00\cdots 0\rangle\langle 11\cdots 1| + |11\cdots 1\rangle\langle 00\cdots 0|.$$

Their eigenstates take the form of generalized GHZ states

$$|\phi_{2,N;k}\rangle = \frac{1}{\sqrt{2}}(|00\cdots 0\rangle + e^{k\pi}|11\cdots 1\rangle), \quad k = 0, 1. \tag{7}$$

It is clear that states (7) manifest complete entanglement. According to Klyachko and Shumovsky (Klyachko and Shumovsky 2004), condition of complete entanglement has the form

$$\forall j, i \quad \langle\psi|X_i^{(j)}|\psi\rangle = 0, \tag{8}$$

providing the maximal total amount of quantum fluctuations. Here $X_i \in \mathcal{L}$ denotes a local basic observable and $j$ labels the party. In the case of qubits, the local basic observables are given by the Pauli operators. Since the action of Pauli operators $\sigma_\alpha$ on a qubit state $|\ell\rangle$ $(\ell = 0, 1)$ either changes $\ell$ (in the case of $\alpha = x, y$) or changes sign (in the case of $\alpha = z$), the states (7) obey the condition (8).

For $N$ qutrits with $n = 3$, the exponential phase operator (6) takes the form

$$E_{3,N} = |00\cdots 0\rangle\langle 11\cdots 1| + |11\cdots 1\rangle\langle 22\cdots 2| + |22\cdots 2\rangle\langle 00\cdots 0|.$$

The corresponding eigenstates have the form

$$|\phi_{3,N;k}\rangle = \frac{1}{\sqrt{3}}(|00\cdots 0\rangle + e^{i\phi_k}|11\cdots 1\rangle + e^{2i\phi_k}|22\cdots 2\rangle), \tag{9}$$

where $\phi_k = 2k\pi/3$, $k = 0, 1, 2$. It is also easily seen that the states (9) obey the condition (8) with local basic observables $X_i$ $(i = 1, \cdots, 8)$, forming an orthogonal basis of the $su(3)$ algebra. States of the form (9) were considered by Shumovsky (Shumovsky 1999) in the context of quantum phase of the angular momentum of photons (at $N = 1$) and by Bechman-Pasquinucci and Peres (Bechman-Pasquinuci and Peres 2000) in connection with realization of quantum ternary logic (at $N = 2$).

Continuing the above process, one can arrive to the conclusion that the generic entangled states of $N$ qunits have the form

$$|\phi_{n,N;k}\rangle = \frac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} e^{i\ell\phi_k}|\ell; N\rangle, \quad \phi_k = \frac{2k\pi}{n}, \tag{10}$$

where

$$|\ell; N\rangle \equiv |\underbrace{\ell\ell\cdots\ell}_{N \text{ times}}\rangle$$

are the "homogeneous" states. The states (10) are eigenstates of the operator (6).

The above construction of the generic entangled states of qunits as the $su(2)$ phase states of dimension $n$ also gives a clue in the definition of physical Hamiltonians, whose eigenstates are completely entangled states. Those Hamiltonians can be defined, for example, as the cosine and sine of the $su(2)$ phase operators

$$C = \frac{E + E^+}{2}, \quad S = \frac{E - E^+}{2i} \tag{11}$$

and as the $su(2)$ phase operator

$$\Phi = \sum_k \phi_k |\phi_{n,N;k}\rangle\langle\phi_{n,N;k}|. \tag{12}$$

At $n = n = 2$ (two qubits), for example, the cosine operator in (11) takes the form

$$C \sim (\sigma_x^{(1)} \otimes \sigma_x^{(2)} - \sigma_y^{(1)} \otimes \sigma_y^{(2)}) \sim (\sigma_+^{(1)} \otimes \sigma_-^{(2)} + \sigma_-^{(1)} \otimes \sigma_+^{(2)}).$$

This operator is similar to the interaction Hamiltonian for a system of two two-level atoms, widely discussing in the context of atomic entanglement (e.g., see Çakır *et al* 2005 and references therein). Being summarized over a huge number of two-level system, it formally coincides with the quasi-spin form of the BCS Hamiltonian in the theory of superconductivity (Bogoliubov *et al* 1994) describing the set of completely entangled Cooper pairs in superconductors.

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